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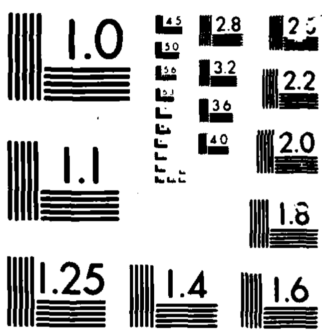

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STEEP GRAVITY WAVES: HAVELOCK'S METHOD REVISITED

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ABSTRACT

Gravity waves propagating at the surface of a fluid of infinite depth are considered. The problem is formulated in terms of a series expansion due to Havelock. The series is truncated after a finite number of terms and the unknown coefficients are found by collocation. It is shown that this simple numerical procedure yields accurate results for waves of arbitrary steepness.

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SIGNIFICANCE AND EXPLANATION

Over the last 15 years many efficient numerical schemes have been developed to compute steep water waves. These schemes are often based on integro-differential equation formulations or on collocation techniques.

In this paper we present a new numerical approach based on an expansion proposed by Havelock in 1919. This scheme is very easy to implement and yields highly accurate results for waves of arbitrary steepness.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STEEP GRAVITY WAVES: HAVELOCK'S METHOD REVISITED

Jean-Marc Vanden-Broeck*

1. Introduction

This paper deals with the numerical computation of periodic two-dimensional gravity waves propagating at the surface of a fluid of infinite depth. This problem was considered before by many investigators. Most of the existing numerical procedures belong to one of two main classes.

In the first class the problem is formulated as an integro-differential equation for the free surface profile. This equation is discretized and solved numerically by Newton's method (see for example Schwartz and Vanden-Broeck[12], Chen and Saffman[2], Vanden-Broeck Schwartz[15], and Vanden-Broeck[14]).

In the second class the solution is represented by a Fourier expansion. The unknown Fourier coefficients are found analytically as series in powers of a parameter equivalent to the wave steepness (Stokes[13], Schwartz[11], Lonquet-Higgins[7], Cokelet[4]) or numerically by collocation (Chen and Saffman[3], Rienecker and Fenton[10]).

Numerical schemes of the second class are usually inefficient to compute directly steep waves because the Fourier coefficients decay too slowly as the wave height approaches its maximum. Accurate solutions can however be obtained indirectly by recasting the Fourier expansion as Padé approximants (Schwartz[11], Lonquet-Higgins[7], Cokelet[4]). On the other hand, very steep waves can be calculated directly by using numerical schemes of the first class (Schwartz and Vanden-Broeck[12], Chen and Saffman[2], Vanden-Broeck and Schwartz[15]).

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Successful numerical procedures of the second class have been developed to compute directly the highest wave (Michell[8], Olfe and Rottman[9], Williams[16]). The basic idea of these numerical procedures is to represent the solution by an expansion which takes into account the fact that the highest wave has a sharp crest with a 120° degree angle.

In this paper we present a numerical scheme of the second class which enables us to compute directly very steep waves. Our procedure follows closely the work of Havelock[6] and includes as a particular case Olfe and Rootman's[9] scheme.

2. Numerical Results

We consider two-dimensional periodic waves of wavelength λ and phase velocity C propagating under the influence of gravity g at the surface of a fluid of infinite depth. We choose a frame of reference in which the waves are steady and we introduce dimensionless variables by taking λ as the unit length and C as the unit velocity. The effects of compressibility, viscosity and surface tension are neglected.

We introduce cartesian coordinates with the x-axis at the mean water level and the y-axis directed vertically upwards. Gravity is acting in the negative y-direction. Next we define the complex potential $f = \varphi + i\psi$ and the complex velocity $W = u - iv$. Here φ is the potential function, ψ the stream function, u the x-component of the velocity and v the y-component of the velocity. Without loss of generality we choose $\psi = 0$ on the free surface and $\varphi = 0$ at one crest.

The condition of constant pressure $(p = 0)$ on the free surface can be written

$$|W|^2 + \frac{4\pi}{\mu} y = 1, \quad \psi = 0 \tag{1}$$

where

$$\mu = \frac{2\pi C^2}{g\lambda} . \tag{2}$$

Following Stokes[13] we seek W as an analytic function of $f = \phi + i\psi$ in the lower half plane $\psi < 0$. This function is periodic and tends to one as $f \to -\infty$. Thus we have

$$W(f + 1) = W(f) \tag{3}$$

$$W \to 1 \quad \text{as} \quad f \to -\infty . \tag{4}$$

We find it convenient to eliminate y from (1) by differentiating (1) with respect to φ. Using the identity

$$\frac{\partial x}{\partial \varphi} + i \frac{\partial y}{\partial \varphi} = W^{-1} \tag{5}$$

we obtain

$$|W| \frac{\partial |W|}{\partial \varphi} - \frac{2\pi}{\mu} \frac{\operatorname{Im} W}{|W|^2} = 0, \qquad \psi = 0 . \tag{6}$$

Following Cokelet[4] we define the amplitude parameter ε^2 by the relation

$$\varepsilon^2 = 1 - |W(0)|^2 |W(1)|^2 . \tag{7}$$

For the highest wave $W(0) = 0$ and $\varepsilon = 1$. In general ε ranges between 0 and 1.

The relations (3) and (4) show that W can be represented by the following expansion:

$$W(f) = 1 + \sum_{n=1}^{\infty} b_n e^{-i2\pi nf} . \tag{8}$$

Because of the symmetry of the wave about $f = 0$, the coefficients b_n are real. They have to be found to satisfy (6) on $\psi = 0$. This can be achieved approximately by using the collocation procedure mentioned in the introduction. Thus we truncate the series in (8) after N terms and we introduce the N mesh points

$$\varphi_I = \frac{2I - 1}{4N} \qquad I = 1,\ldots,N . \tag{9}$$

Using (8) we obtain $W(\varphi_I)$ in terms the coefficients b_n. Substituting these expressions into (6) we obtain N nonlinear algebraic equations for the $N + 1$ unknowns $\mu, b_1, \ldots, b_N$. Another equation is obtained by using (7) where ε^2 is specified. This sytem of $N + 1$ equations is solved by Newton's method. Once the coefficient b_n are found, the free surface profile can be obtained by integrating numerically (5).

In Table I we present numerical values of μ versus ε^2 obtained with $N = 60$. For comparison we also show the accurate values of μ obtained by Cokelet[4]. Our values agree with those of Cokelet to 5 decimal places for $\varepsilon^2 < 0.6$. However, the accuracy of our results decreases rapidly as ε^2 approaches 1. This is due to the slow convergence of the expansion (8) as the wave of maximum height is approached.

The highest wave, (i.e. $\varepsilon^2 = 1$) is characterized by a corner at the crest with an enclosed angle of 120° (Stokes[13], Amick et al.[1]). Therefore

$$W(f) \sim f^{1/3} \quad \text{as} \quad f \to 0 . \tag{10}$$

Following Michell[8] and Olfe and Rootman[9] we compute the highest wave by replacing (8) by

$$W(f) = (1 - e^{-2i\pi f})^{1/3} \left(1 + \sum_{n=1}^{\infty} C_n e^{-2i\pi nf}\right) . \tag{11}$$

Table I: Values of μ for $0.6 < \epsilon^2 < 0.99$ obtained by using (8).

ϵ^2	N = 60	Cokelet
0.6	1.12229	1.12229
0.8	1.17209	1.17093
0.9	1.20088	1.19014
0.94	1.21684	1.19404
0.99	1.25358	1.19329

The expansion (11) satisfies (10). We truncate the expansion (11) after $N - 1$ terms and satisfy (6) at the N mesh points (9). This yields N equations for the N unknowns $\mu, C_1, \ldots, C_{N-1}$. This system was first solved by Olfe and Rootman[9]. In particular they found $\mu = 1.93072$. We have repeated the calculation and confirmed this value.

The previous considerations suggest to combine the advantages of (8) and (11) by representing the solution by the expansion

$$W(f) = (1 - \beta e^{-i2\pi f})^{1/3}(1 + \sum_{n=0}^{\infty} d_n e^{-2i\pi nf}) . \qquad (12)$$

This expansion was first proposed by Havelock[6]. As $\epsilon \to 0$, $\beta \to 0$ and (12) approaches (8). Furthermore $\beta \to 1$ as $\epsilon \to 1$, so that (12) includes (11) as a particular case.

We now truncate (12) after $N - 1$ terms and satisfy (6) at the N mesh points (9). Thus we obtain N equations for the $N + 1$ unknowns $\beta, \mu, d_1, \ldots, d_{N-1}$. The last equation is given by (7) where ϵ^2 is specified.

Numerical values of μ versus ϵ^2 for $N = 60$, 80 and 120 are presented in Table II. The values obtained by Cokelet[4] are also shown in the table. These results indicate that the scheme converges as N increases. Furthermore, the procedure yields values as accurate as those of Cokelet[4] for values of ϵ close to one. A comparison between the values for $N = 60$ in Tables I and II, show clearly that the expansion (12) converges much faster than the expansion (8).

Table II: Values of μ for $0.6 < \varepsilon^2 < 0.99$ obtained by using (12).

ε^2	N = 60	N = 80	N = 120	Cokelet
0.6	1.12229	1.12229	1.12229	1.12229
0.8	1.17096	1.17094	1.17093	1.17093
0.9	1.19007	1.19025	1.19019	1.19014
0.94	1.19310	1.19367	1.19409	1.19404
0.99	1.19321	1.19324	1.19332	1.19329

It is worthwhile mentioning that Grant[5] and Schwartz[11] have demonstrated that the Havelock expansion (12) produces the wrong type of singularities above the fluid (i.e. in $\psi > 0$). This does not of course invalidate the Havelock expansion. In fact, our numerical results show that this expansion is rapidly convergent inside the fluid and on the free surface (i.e. in $\psi < 0$).

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